university of groningen

**Group Theory Resit Exam**

Date: 01 February 2024

Place: Exam Hall 1, Aletta Jacobs Hall

Time: 8:30 – 10:30

## Instructions

- Clearly write your name and student number on each sheet.
- To get full points, you must provide complete arguments and computations. You will get no points if you do not explain your answer.
- While solving a problem, you can use any statement that needs to be proved as a part of another problem even if you did not manage to prove it; e.g. you can use part (a) while solving part (b) even if you did not prove (a).
- The exam consists of 5 questions. You can score up to 36 points and you get 4 points for free. This way you will score in total between 4 and 40 points.

## Problems

**1** Let $\sigma = (1\,2)(1\,3\,4)(2\,4\,5) \in S_5$.

(a) [**2 points**] Determine the order of $\sigma$.

(b) [**2 points**] Compute $\sigma^{2024}$.

(c) [**2 points**] Decide if $\sigma$ is even or odd.

(d) [**3 points**] Show that all elements of order 14 in $S_9$ are conjugate.

**2** Fix $n \geq 1$ and let

$$O(n) \coloneqq \{A \in \mathrm{GL}_n(\mathbb{R}) \mid A^T A = I\},$$
$$SO(n) \coloneqq \{A \in O(n) \mid \det(A) = 1\}.$$

(a) [**3 points**] Show that $O(n)$ is a subgroup of $\mathrm{GL}_n(\mathbb{R})$.

(b) [**4 points**] Show that $SO(n)$ is a normal subgroup of $O(n)$ and that there is an isomorphism

$$O(n)/SO(n) \cong \{\pm 1\}.$$

Here, $\{\pm 1\}$ denotes the subgroup of $\mathbb{R}^\times$ whose elements are 1 and $-1$.

**3** [**5 points**] Show that there are no simple groups of order 56.

**4** (a) [**3 points**] Let $H$ be the subgroup of $\mathbb{Z}^3$ with basis $g_1 = (1, -2, -8)$, $g_2 = (-1, 20, -1)$, $g_3 = (2, -4, 7)$. Determine the rank and elementary divisors of $\mathbb{Z}^3/H$.

(b) [**3 points**] Find all abelian groups of order $162 = 2 \cdot 3^4$ up to isomorphism.

**5** **Prove/Disprove.** For each of the following statements, prove the statement if it is true and disprove it if it is false.

(a) **[3 points]** There exists a transitive action of the dihedral group $D_3$ on a set of cardinality 4.

(b) **[3 points]** The action of the alternating group $A_4$ on $\{1, 2, 3, 4\}$ is fixpoint free.

(c) **[3 points]** If $G$ is a finite group and $H_1, H_2 \leq G$ are two subgroups of the same cardinality, then $H_1$ and $H_2$ are conjugate.

GOOD LUCK! ☺